Chapter

2

Motion of centre of mass

2.1 Velocity of centre of mass of a system

In chapter 1, we know how to locate the centre of mass of a system. Now we turn our attention to the velocity of the centre of mass of a system. Equation 1.2(iii) obtained in section 1.2 of chapter 1 is for position of centre of mass of n particles relative to the origin O of a given reference frame. When the centre of mass of a system is not stationary in a given reference frame, its velocity can be obtained from the time derivative of the equation for the position of centre of mass.

According to equation 1.2(iii),

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{M}$$

We obtain the centre of mass velocity V_{cm} by differentiating each term in equation 1.2 (iii) with respect to time

$$\frac{d\vec{r}_{cm}}{dt} = \frac{m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots + m_n\frac{d\vec{r}_n}{dt}}{M}$$

or $$\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n}{M} \qquad 2.1(i)$$

or $$\vec{V}_{cm} = \frac{\sum_{i}^{n} m_i\vec{v}_l}{M}$$

$\therefore$ $$M\vec{V}_{cm} = \sum_{i=1}^{n} m_i\vec{V}_l \qquad (2.1(ii))$$

The quantity $m_i v_i$ is the momentum of i[th] particle. Thus the right hand side of equation 2.1(ii) is the total momentum of the system of n particles. The left hand side of this equation is the momentum of a particle with mass equal to total mass M of the system and velocity equal to the velocity of centre of mass of the system. **We therefore have the important result that the total momentum of a system of particles is equal to the momentum of a single particle of mass M located at the centre of mass of the system and is moving with centre of mass velocity V_{cm}.**

Thus, momentum of a system is

$$\vec{P} = M\vec{V}_{cm} \qquad (2.1(iii))$$

We are generally interested to calculate the momentum of rigid bodies like billiard balls or vehicles. The momentum of rigid body is easily found by product of mass of the body and velocity of the centre of mass of the body.

Example 2.1 Velocity of centre of mass of a system of two vehicles

A bus of mass 3000 kg is moving towards east with a velocity of 15 m/s on a straight road. A car of mass 1000 kg is moving towards west with a velocity of 10 m/s. Find velocity of centre of mass of the two vehicle system.

Solution To determine the velocity of centre of mass of bus plus car system, total mass m_1 of bus can be assumed as a particle moving with its velocity. Similarly, the total mass m_2 of the car can be assumed as a particle moving with its velocity. The velocity of centre of mass is

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{3000 \times 15 - 1000 \times 10}{3000 + 1000}$$

$$= 8.75 \text{m/s}.$$

Thus, the centre of mass of the two vehicles system is moving with a velocity of 8.75 m/s towards east.

Example 2.2 **Velocity of centre of mass of a system of particles moving in different directions**

A 5 kg particle is moving in north direction with a velocity 3 m/s. A 2 kg particle is moving with a velocity 15 m/s in a direction 37° west of south. A third particle of mass 3 kg is moving such that the centre of mass of a system of these three particle moves towards north east direction with a velocity of $5\sqrt{2}$ m/s.

(A) Find momentum of the system

(B) Find the velocity of third particle.

Solution The momentum of system is given by product of total mass of the system and centre of mass velocity.

$\therefore$ Thus, the total momentum of system is

$$P = (m_1 + m_2 + m_3)\vec{v}_{cm}$$

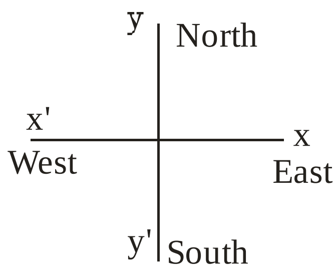
The magnitude of momentum of the system is

$$P = (5 + 2 + 3)5\sqrt{2}$$

$$= 50\sqrt{2} kg \ m/s.$$

Thus, the momentum of the system is 50 kg m/s in north east direction.

(B) Take the axes as shown in **figure 2.2(A)**.



(Figure 2.2(A))

The velocities of three particles are as follow:

Particle	Mass (in kg)	Velocity (in m/s)
First	5	$3\hat{j}$
Second	2	$-15\sin 37\hat{i} - 15\cos 37\hat{j}$
Third	3	$\vec{V}_3$

The velocity of centre of mass of system is

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1 + m_2 + m_3}$$

or $$5\sqrt{2}\cos 45\hat{i} + 5\sqrt{2}\sin 45\hat{j} = \frac{5\left(3\hat{j}\right) + 2\left(-15\sin 37\hat{i} - 15\cos 37\hat{j} + \vec{v}_3\right)}{5+2+3}$$

or $$\left(5\hat{i} + 5\hat{j}\right)10 = 15\hat{j} - 30\times\frac{3}{5}\hat{i} - 15\times\frac{4}{5}\hat{j} + \vec{v}_3$$

$\therefore$ $$\vec{v}_3 = (50+18)\hat{i} + (50 - 15 + 12)\hat{j} = \left(68\hat{i} + 47\hat{j}\right)m/s$$

$\therefore$ $$v_3 = \sqrt{(68)^2 + (47)^2}$$

$$= 82.66 \text{ m/s}$$

Let the velocity of third particle makes a angle with east direction.

$\because$ $$\tan - = \frac{v_y}{v_x} = \frac{47}{68}$$

$\therefore$ $$- = \tan^{-1}\left(\frac{47}{68}\right).$$

Thus, the third particle is moving in a direction $\tan^{-1}\left(\frac{47}{68}\right)$ north of east with a velocity of 82.66 m/s.

Example 2.3 Velocity of centre of mass of particles in three dimensional motions

Particles A, B, C and D having masses 1 kg, 2 kg, 4 kg and 8 kg respectively are moving with velocities $\left(\hat{i}+5\hat{j}-3\hat{k}\right)$m/s, $\left(3\hat{i}+3\hat{j}-\hat{k}\right)$m/s, $\left(-7\hat{i}+\hat{k}\right)$ m/s and $\left(\hat{i}+\hat{j}\right)$m/s respectively. Find velocity of centre of mass of the system of the particles.

Solution The velocity of centre of the system is

$$\vec{v}_{cm}=\frac{m_A\vec{v}_A+m_B\vec{v}_B+m_C\vec{v}_C+m_D\vec{v}_D}{m_A+m_B+m_C+m_D}$$

$$=\frac{1\left(\hat{i}+5\hat{j}-3\hat{k}\right)+2\left(3\hat{i}+3\hat{j}-\hat{k}\right)+4\left(-7\hat{i}+\hat{k}\right)+8\left(\hat{i}+\hat{j}\right)}{1+2+4+8}$$

$$=\left(-\frac{13\hat{i}}{15}+\frac{19\hat{j}}{15}-\frac{1}{15}\hat{k}\right)m/s$$

2.2 Acceleration of centre of mass of a system

In section 2.1, we know how to find the velocity of centre of a system. Now we turn our attention to the acceleration of centre of mass of a system. We can find the acceleration of centre of mass by again differenting each term in equation 2.1(i) with respect to time.

According to equation 2.1(i),

$$\vec{V}_{cm}=\frac{m_1\vec{v}_1+m_2\vec{v}_2+...+m_n\vec{v}_n}{M}$$

or $$\frac{d\vec{v}_{cm}}{dt} = \frac{m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} + ... + m_n \frac{d\vec{v}_n}{dt}}{M}$$

$\therefore$ $$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ... + m_n\vec{a}_n}{M} \qquad 2.2(i)$$

Example 2.4 Acceleration of centre of mass of a vehicles system

Two cars A and B having masses 1000 kg and 2000 kg respectively are moving in same direction. At $t = t_1$, car A starts to accelerate with an acceleration 10 m/s^2 while brake is applied by driver of car B which is cause of deceleration of 5 m/s^2. Find acceleration of centre of mass of the system of the cars at $t = t_1$.

Solution The acceleration of centre of mass system is

$$a_{cm} = \frac{m_A a_A + m_B a_B}{m_A + m_B}$$

$$= \frac{1000 \times 10 + 2000(-5)}{1000 + 200}$$

$$= 0$$

Thus, the acceleration of the centre of mass of the system of cars is zero

Example 2.5 Acceleration of centre of mass of a system of particles in three dimensional motions

A system consists of masses 1 kg, 2 kg and 4 kg moving with acceleration $\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) m/s^2, \left(2\hat{i} - \hat{j} + 3\hat{k}\right) m/s^2$ and $\left(\hat{i} + \hat{k}\right) m/s^2$ respectively at t = 0. Find acceleration of centre of mass of the system at t = 0.

Solution The acceleration of the centre of mass of the system is

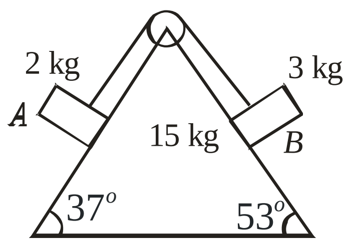
$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3}{m_1 + m_2 + m_3}$$

$$= \frac{1\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + 2\left(2\hat{i} - \hat{j} + 3\hat{k}\right) + 4\left(\hat{i} + \hat{k}\right)}{1 + 2 + 4}$$

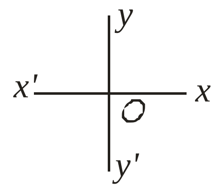
$$= \frac{9\hat{i} + 13\hat{k}}{7}$$

$$= \left(\frac{9}{7}\hat{i} + \frac{13}{7}\hat{k}\right) m/s^2$$

Example 2.6 Consider the situation shown in figure. Both the pulley and the string are light. Blocks A and B are smooth while wedge of mass 15 kg is fixed. Find acceleration of centre of mass of wedge plus blocks system shown in **figure2.6 (A)**.

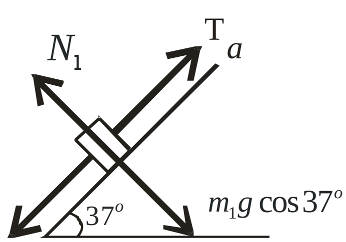


(Figure 2.6(A)



(Figure 2.6(B)

Solution The force diagram of block A is shown in **figure 2.6(C)**. According to Newton's second law of motion,

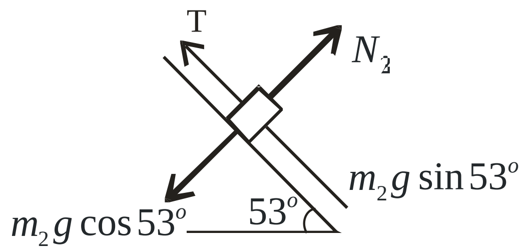


(Figure 2.6(C)

$$T - m_1 g \sin 37 = m_1 a$$

or $$T - 12 = 2a \qquad \text{(i)}$$

The force diagram of block B is shown in **figure 2.6(D)**.



(Figure 2.6(D))

According to Newton's second law of motion,

$$m_2 g\sin 53^o - T = m_2 a$$

or $$24 - T = 3a \qquad \text{(ii)}$$

Adding eq^n (i) and (ii)

$$24 - 12 = 5a$$

$\therefore$ $$a = 2.4 \text{ m/s}^2.$$

Take the axes shown in figure 2.6(B). The acceleration of the block A is

$$\vec{a}_1 = 2.4\cos 37^o\,\hat{i} + 2.4\sin 37^o\,\hat{j}$$

Similarly, the acceleration of the block B is

$$\vec{a}_2 = 2.4\cos 53^o\,\hat{i} - 2.4\sin 53^o\,\hat{j}.$$

As the wedge is fixed, its acceleration is zero. The acceleration of centre of mass of the system is

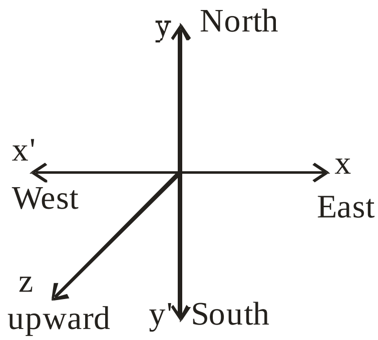
$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3}{m_1 + m_2 + m_3}$$

$$= \frac{4.8\cos 37\hat{i} + 4.8\sin 37\hat{j} + 7.2\cos 53\hat{i} - 7.2\sin 53\hat{j} + \vec{0}}{2+3+15}$$

$$=\left(0.41\hat{i}-0.14\hat{j}\right)m/s^2$$

Example 2.7 Three particles A, B and C having masses 1 kg, 2 kg and 3 kg are being acted upon by forces 1N in east direction, 2 N in north direction and 3 N in upward direction respectively. Find the magnitude of acceleration of centre of mass of the system.

Solution Take the axes as shown in **figure 2.7(A)**.



(Figure 2.7(A))

The acceleration of particle A is

$$\vec{a}_1=\left(\frac{1}{1}\right)\hat{i}=\hat{i}$$

The acceleration of particle B is

$$\vec{a}_2=\left(\frac{2}{2}\right)\hat{j}=\hat{j}$$

The acceleration of particle C is

$$\vec{a}_3=\left(\frac{3}{3}\right)\hat{k}=\hat{k}$$

The acceleration of the centre of mass of system is

$$\vec{a}_{cm}=\frac{m_1\vec{a}_1+m_2\vec{a}_2+m_3\vec{a}_3}{m_1+m_2+m_3}$$

$$= \left(\frac{1}{6}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{2}\hat{k}\right) m/s^2$$

The magnitude of the acceleration of the centre of mass of the system is

$$a_{cm} = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= 0.62 \text{ m/s}^2$$

2.3 Newton's second law of motion and the centre of mass

Consider two particles A and B having masses m_1 and m_2 are connected to a light spring of spring constant k. They rest on a smooth horizontal surface (shown in **figure 2.3(a)**).

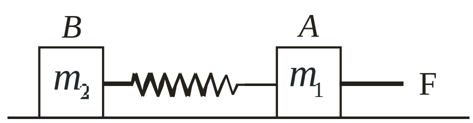


Figure 2.3(a)

At t = 0, a force F is applied on the particle A. Let at an instant t, the extension in spring is x.

The acceleration of particles A and B at instant t are $a_1 = \frac{F - kx}{m_1}$ and $a_2 = \frac{kx}{m_2}$. Taking advantage of equation 2.2(i), the acceleration of centre of mass of the system of two particles is

$$a_{cm} = \frac{m_1\left(\frac{F - kx}{m_1}\right) + m_2\left(\frac{kx}{m_2}\right)}{m_1 + m_2}$$

$$= \frac{F}{m_1 + m_2}. \qquad (2.3(i))$$

For extensive discussion of our two particles system, it is essential to examine the forces acting on the particles. The internal force on the system is that exerted by the spring. The external forces on the system are the normal reaction exerted by the surface on the particles, the force of gravity exerted on each particle, which balances the normal reaction and the applied force F on the particle A. The resultant force on the system is found by vector sum of external forces and internal force p. The force exerted by the spring on particle A is equal and apposite to that exerted on the particle B. From this it is clear that the sum of internal forces in a system is always zero. The sum of the forces on the system can be replaced by the sum of all external forces. The resultant external force on our system is applied force F on the particle A. So the equation 2.3(i) is restated here, rearranged slightly for convenience.

$$F = (m_1 + m_2)a_{cm} \qquad (2.3(ii)$$

This result of course, is Newton’s second law of motion. This remarkable equation says that no matter how complicated the system of particles, the centre of mass moves like an imaginary particle of mass equal to total mass M of the system under the influence of the net external force on the system. In general, for a system of n particles,

$$\sum \vec{F}ext = M\vec{a}_{cm} \qquad (2.3(iii)$$

Where vector sum of external forces on the system is $\sum \mathbf{Fext}$ and M is the total mass of the system

2.4 Stationary centre of mass of a system of moving particles

It is clear from our arguments about motion of centre of mass of a system that the centre of mass moves like a particle of mass equal total mass of system located at the system centre of mass and resultant external force acting on the system is applied on it. In this case the acceleration of centre of mass is quite independent of the points to which external forces are applied.

Let us consider a system of n particles. Each particle is acted upon by forces due to all other particles of the system while net external force on the system is zero.

It follows from eq^n (2.3(iii) that if $\vec{\mathbf{F}}\mathbf{ext} = 0$, then $\vec{a}_{cm} = 0$, and therefore, velocity of centre of mass is constant. If we assume that the particles of the system are moving in an inertial frame K such that the centre of mass of the system is initially at rest in the inertial frame K The velocity of centre of mass at an instant t is

$$V_{cm} = u_{cm} + a_{cm} t$$

$$= 0 + 0 \times t = 0.$$

From this it is important to keep in mind that if net external force on system is zero and its centre of mass is initially at rest in an inertial reference frame K, it remains fixed in the inertial reference frame K even when the particles individually move and accelerate.

Taking the advantage eq^n 2.1(i), the momentum of the considered system is always zero.

$\because$ $$\vec{V}_{cm} = \frac{m_1 v_1 + m_2 v_2 + ... + m_n \vec{v}_n}{m_1 + m_2 + ... m_n}$$

or $$0 = m_1 \vec{v}_1 + m_2 \vec{v}_2 + ... + m_n \vec{v}_n$$

or $$m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + ... + m_n \frac{d\vec{r}_n}{dt} = \vec{0}$$

or $$m_1 \Delta\vec{r}_1 + m_2 \Delta\vec{r}_2 + ... + m_n \Delta\vec{r}_n = 0 \qquad 2.4(i)$$

The relative displacement of ith particle seen from first particle is given by

$$\Delta\vec{r}_i = \Delta\vec{r}_i - \Delta\vec{r}_1$$

$\because$ $$m_1\Delta\vec{r}_1 + m_2\Delta\vec{r}_2 + ... + m_n\Delta\vec{r}_n = \vec{0}$$

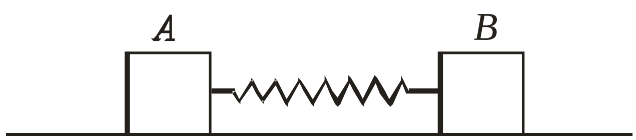
or $$m_1\left(\Delta\vec{r}_1 - \Delta\vec{r}_1\right) + m_2\left(\Delta\vec{r}_2 - \Delta\vec{r}_1\right) + ...m_n\left(\Delta\vec{r}_n - \Delta\vec{r}_1\right)$$

$$= -\left(m_1\Delta\vec{r}_1 + m_2\Delta\vec{r}_1 + ... + m_n\Delta\vec{r}_n\right)$$

or $$\sum_{i=2}^{n} m_i\Delta r_i = -\Delta\vec{r}_1\sum_{i=1}^{n} m_i$$

$\therefore$ $$\Delta\vec{r}_1 = -\frac{\sum_{i=2}^{n} m_i\Delta\vec{r}_{i1}}{\sum_{i=1}^{n} m_i} \qquad 2.4(ii)$$

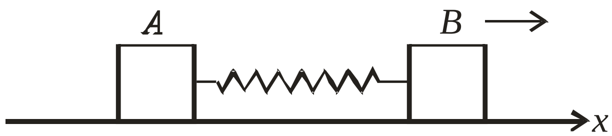
2.8 Two blocks A and B of mass 2 kg and 3 kg respectively are tied by a light spring of spring constant 100 N/m (shown in figure 2.8(A)) and placed on a smooth horizontal surface. Blocks are released from rest at the instant when compression in spring is 10 cm. Find distance moved by the blocks before spring comes in its natural length.



(Figure 2.8(A))

Solution Consider block A plus block B to be a system. As both blocks are released from rest, the initial velocity of centre of mass of the system is zero. The forces exerted by spring on blocks are internal force of the system. No external force acts on the system along the horizontal surface. So acceleration of centre of mass the system is zero. Hence, the centre mass of the system remains fixed during motion of blocks. When spring comes in natural length, the relative displacement of block B seen from block A is

$$\vec{r}_{21} = (0.1m)\hat{i}.$$



The displacement of block A is

$$\Delta\vec{r}_1 = -\frac{m_2\Delta\vec{r}_{21}}{m_1+m_2} \qquad \text{(form eq}^n\text{2.4 (ii))}$$

$$= -\frac{3\times 0.1\hat{i}}{3+2}$$

$$= -\frac{0.3}{5} = -0.06m$$

= - 6 cm

Similarly, $$\Delta\vec{r}_2 = -\frac{m_1\Delta\vec{r}_{12}}{m_1+m_2}$$

$$= -\frac{2}{2+3}\left(-0.1\hat{i}\right)$$

= 0.04 m

= 4 cm

Thus, the block A moves 6 cm leftward along the horizontal surface and the block B moves 4 cm righter along the horizontal surface.

Alternative method:

Let the magnitude of displacement of blocks A and B are Dr_1 and Dr_2 along the surface

$\because$ $$\Delta r_1 + \Delta r_2 = 10\ cm \qquad \ldots\text{(i)}$$

$$\Delta\vec{r}_{cm} = -\frac{m_1\Delta r_1 + m_2\Delta r_2}{m_1 + m_2}$$

or $$0 = -m_1\Delta r_1 + m_2\Delta r_2$$

or $$m_1\Delta r_1 = m_2 r_2$$

or $$2\Delta r_1 = 3\Delta r_2 \qquad \ldots\text{(ii)}$$

From eq^n (i) and (ii) $h = \frac{{}^{3}{}_{T}^{2}}{2g} = \frac{(3.55)^2}{2(9.8)}$

$$\Delta r_1 = 6\ cm\ ,\quad \Delta r_2 = 4\ cm$$

2.9 A bomb of mass 10 kg initially at rest on a smooth horizontal surface explodes and breaks into four fragments of mass 1 kg, 2 kg, 3 kg and 4 kg. 1 kg fragment moves in east direction with a velocity 5 m/s, 2 kg fragment moves in north direction with velocity 10 m/s, and 3 kg fragment moves in north east direction with velocity $20\sqrt{2}$ m/s. What is the velocity of 4kg fragment?

Solution As bomb is initially at rest, the velocity of centre of mass is zero before explosion. During explosion, no external force acts on the system along the surface. This means that the centre of mass remains stationary.

Take the axes as shown in fig. 2.9(A).

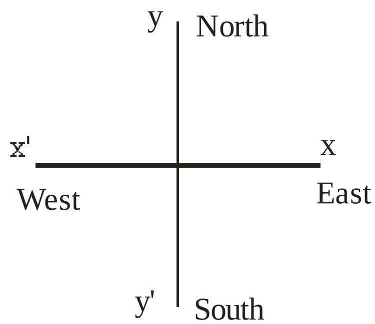


Figure 2.9(A)

Fragment	mass (kg)	velocity (in m/s)
First	1	05
Second	2	$10\,\hat{i}$
Third	3	$10\hat{i} + 10\hat{j}$
Forth	4	$\vec{V}_4$

Taking the advantage of eq^n 2.1(i),

$$\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + m_4\vec{v}_4}{m_1 + m_2 + m_3 + m_4}$$

or $$\vec{o} = 1\left(5\hat{i}\right) + 2\left(10\,\hat{j}\right) + 3\left(10\hat{i} \text{ - } 10\,\hat{j}\right) + 4\vec{v}_4$$

∴ $$\vec{v}_4 = \frac{25}{4}\hat{i} + \frac{10}{4}\hat{j}$$

Let the velocity of forth fragment makes θ angle with east direction.

∴ $$\tan - = \frac{10}{4\times\frac{25}{4}} = \frac{2}{5} = 0.4$$

∴ $$- = \tan^{-1}(0.4)$$

The magnitude of velocity of forth fragment is

$$v_4 = \sqrt{\left(\frac{25}{4}\right)^2 + \left(\frac{10}{4}\right)^2}$$

$$= 26.9 \text{ m/s}$$

Thus, the velocity of forth fragment is 26.9 m/s in a direction $\tan^{-1}$ (0.4) north of east.

2.10 A plank of mass 40 kg and length 9.4 m is placed on a smooth horizontal surface. At the left end is a boy named Rishu of mass 30 kg holding a ball of mass 4 kg. At the right end is a boy named Priyansh of mass 20 kg. At t = 0, Rishu throws the ball toward Priyansh at a speed v_0 = 9 m/s.

(A) What is magnitude of the velocity 'V' of the plank together with boys when the ball is in air?

(B) How long does ball take to reach Priyansh?

(C) During this time how much has the boat moved? (Neglect air resistance)

Solution If the plank, ball, Rishu and Priyansh are taken as a system, net external force on the system in horizontal direction is zero. As initial velocity of the centre of mass is zero, the centre of mass of the system will remain stationary during the motion.

(A) The velocity of centre of mass is zero.

$$\therefore \qquad m_1V + m_2V + m_3V + m_4V_o = 0$$

or

$$(m_1 + m_2 + m_3)V = - m_4V_0$$

$$\therefore \qquad V = - \frac{m_4V_0}{m_1 + m_2 + m_3}$$

$$= - \frac{4v_0}{90} = - \frac{-4 \times 9}{90}$$

$$= - 0.4 \text{ m/s.}$$

Thus, the plank is moving to the left with a speed V = 0.4 m/s.

(B) The time it taken the ball to reach Priyansh will be the time in which it taken the ball and the plank to go a combined distance of L with the ball moving to the right and the plank moving to the left.

Here, $L = V_0 t + Vt$

$= 9t + 0.4t$

$= (9.4)t$

$\therefore \quad t = \frac{L}{9.4} = \frac{9.4}{9.4} = 1\text{sec}.$

(C) During this time, the plank moved a distance

$$d = V\,t = 0.4 \times 1 = 0.4 \text{ m}$$

2.11 A child of mass m_C = 45 kg sitting on horizontal smooth surface, throws a ball of mass m_b = 5 kg from zero height at speed $v_b = 3\sqrt{10} m/s$. The total distance between the ball and the child is x_0 = 10 m when the ball lands.

(a) At what angle was the ball thrown?

(b) How fast is the child moving?

Solution Since there are no external forces in the horizontal direction on the ball – child system, the centre of mass does not move in horizontal direction when the ball is in the air. We can use this fact to find how far the ball traveled before it hit the ground.

Taking the advantage of eq^n 2.4(ii), the horizontal distance moved by ball is

$$x_b = -\frac{m_c x_{cb}}{m_b + m_c}$$

$$= \frac{-45(-10)}{50} = 9m$$

From the formula of range,

$$x_b = \frac{u^2 \sin 2a}{g}$$

or $$9 = \frac{9}{10}\sin 2a$$

$\therefore$ $$\sin 2a = 1$$

$\Rightarrow$ $$a = 45^o$$

Thus, the ball was thrown at an angle 45^o with horizontal.

(b) The velocity of centre of mass is zero in horizontal direction.

$\therefore$ $$0 = \frac{m_b v_b \cos a - m_c v_c}{m_b + m_c}$$

$\therefore$ $$v_c = \frac{m_b v_b \cos a}{m_c}$$

$$= \frac{5 \times 3\sqrt{10}\cos 45}{45}$$

$$= \frac{\sqrt{10}}{3\sqrt{2}} = \frac{\sqrt{5}}{3} = 0.74 m/s$$

2.12 A man of mass m_1 = 50 kg is standing on a stationary flatboat of mass m_2 = 100 kg so that he is 40 m from the shore of a pond. The surface of the pond is smooth. The man walks 9 m on boat away from the shore and then halts. How far is he from the shore at the ends of this time?

Solution In this case, no external force acts on the man-boat system.

$\therefore$ The distance moved by boat on the surface of pond is

$$\Delta \vec{r}_2 = -\frac{m_1 \Delta \vec{r}_{12}}{m_1 + m_2}$$

$$= -\frac{50 \times 9}{50 + 100} = -3m.$$

Negative sign indicates that, the boat moves 3 m towards shore.

Thus, the distance of man from shore is

$$S = 40 + 9 - 3$$

$$= 46 \text{ m.}$$

Alternative method:

In this case, no external force acts on the system in horizontal direction and the initial velocity of centre of mass of the system is zero. Therefore, the centre of mass will remain stationary during motion.

Let boat moves through distance x towards the shore. Therefore, the distance moves by man is 9 – x away from the shore

$\therefore$ $$\Delta \vec{r}_{cm} = \frac{m_1 \Delta \vec{r}_1 + m_2 \Delta \vec{r}_2}{m_1 + m_2}$$

or $$= 50(9 - x) - 100\ x$$

$\therefore$ $$x = \frac{450}{150} = 3\ m$$

Thus distance of man from shore is

$$S = 40 + 9 - x$$

$$= 46 \text{ m.}$$

2.13 An actor of mass m_1 = 120 kg and an actress of mass m_2 = 30 kg are sitting on a smooth ice surface at distance in apart. They starts to pull at what distance from the actor’s original position do they great to each other

Solution Consider actor plus actress to be a system. The force exerted by string on the actor is equal and opposite to that exerted string on the actress.

Thus, resultant force on the system is size. Initially they are in rest. As initially they are at rest, the centre of mass of the system will remain stationary through the problem.

Taking the advantage of eq^n 2.4 (ii)

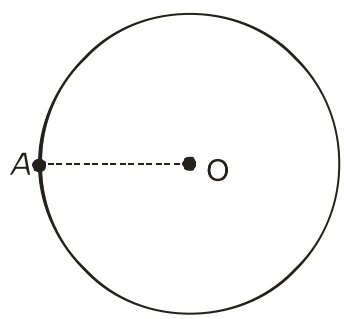
$$\Delta\vec{r}_1 = -\frac{m_2 \Delta\vec{r}_{21}}{m_1 + m_2}$$

$$\therefore \qquad \Delta r_1 = -\frac{m_2(-d)}{m_1 + m_2}$$

$$= -\frac{60}{180}(-9)$$

$$= 3 \text{ m}$$

2.14 An ant of mass m crawls along the periphery of a circular uniform disc of radius R having mass M placed on the smooth horizontal surface. Initially both are at rest. What is the total distance traveled by the centre of disc when the ant reaches at opposite end?



(Figure 2.14(A))

Solution Consider disc plus ant to be a system. As both are initially at rest, the initial velocity of the centre of mass is zero. No external force acts on the system along the horizontal surface. Therefore the acceleration of centre of mass of the system is zero. Hence, the centre of mass remains fixed during motion of ant and disc.

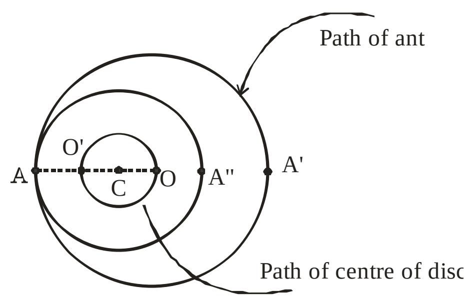
The distance of centre of mass C of the system from centre O of the disc is

$$r_1 = OC = \frac{mR}{m+M}.$$

The distance of centre of mass C of the system from the ant is

$$r_2 = AC = \frac{MR}{m+M}.$$

Since, these distance are constant, the centre of disc and will move of radius OC on circular path about mass centre C. Similarly the ant will describe a circle with radius AC (shown in **Figure 2.14(B)**).



(Figure 2.14(B))

In figure 2.14(B), A is initial position of the ant with respect to disc and A' is final position of ant with respect to disc. A'' is final position of the ant in the frame of ground O' is final position of the disc centre in the frame of ground.

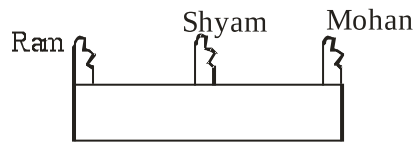
Thus, the distance moved by the centre of the disc is

$$S = \circledR r_1 = \frac{\circledR mR}{m+M}$$

2.15 Ram (60 kg), Sham (80 kg) and Mohan (110 kg) are seated in line on a 9 m long plank (150 kg) at equal distance between them. Ram and

Shyam interchanges their positions while Shyam comes at middle of plank.

How much distance the plank moves?



Solution Here $m_1 = 150$ kg,

$m_2 = 60$ kg, $r_{21} = 4.5$ m

$m_3 = 80$ kg, $r_{31} = -4.5$ m

$m_4 = 110$ kg, $r_{42} = -4.5$ m

The distance moved by the plank is

$$x_1 = -\frac{(m_2 r_{21} + m_3 r_{31} + m_4 r_{41})}{m_1 + m_2 + m_3 + m_4}$$

$$= -\frac{\{60(4.5) + 80(-4.5) + 110(-4.5)\}}{150 + 60 + 80 + 110}$$

$= 1.46$ m

Alternative method

Consider Ram, Shyam, Mohan and plank as a system. As there is no external force on the system and the system is initially at rest, the centre of mass of the system remains fixed.

Let the plank move leftward a distance Δx_1 in the frame of ground. The displacement of Ram in the frame of ground is

$$\Delta x_2 = 4.5 - \Delta x_1$$

The displacement of Shyam in the frame of ground is

$$\Delta x_3 = -(4.5 + \Delta x_1)$$

The displacement of Mohan in the frame of ground is

$$\Delta x_4 = -(4.5 + \Delta x_1).$$

$\because$ $$\Delta x_{cm} = \frac{m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m_4\Delta x_4}{m_1 + m_2 + m_3 + m_4}$$

or $$0 = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 + m_4\Delta x_4$$

or $$0 = 150(-\Delta x_1) + 60(4.5 - \Delta x_1) - 80(4.5 + \Delta x_1) - 110(4.5 + \Delta x_1)$$

$\therefore$ $$\Delta x_1 = 1.46\ m$$

2.16 Three bugs A, B and C having masses m, 2m and 3m respectively are on a rectangular plate of mass 4m placed on smooth horizontal surface. They start to move with velocities v_0, $2v_0$ and $3v_0$ respectively with respect to plate in directions shown in **figure 2.16(A)**.

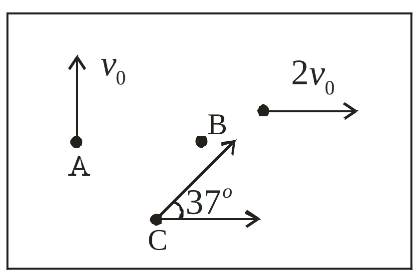


Figure 2.16(A)

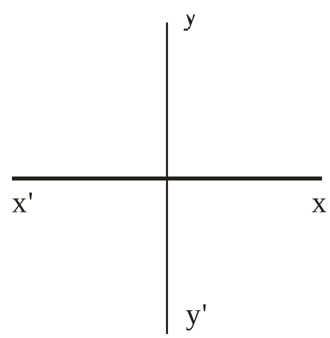


Figure 2.16(B)

What is the magnitude of velocity of the plank?

Solution If the plank + bugs are taken as a system, net external force on the system in x – y plane is zero. As initial velocity of the centre of mass is zero the centre of mass will remain stationary during motion of bugs and plate.

The velocity of centre of mass is zero.

$\therefore$
$$m_1\vec{v}_1 + m_A\vec{v}_A + m_B\vec{v}_B + m_C\vec{v}_C = \vec{0}$$

or
$$m_1(\vec{v}_1 - \vec{v}_1) + m_A(\vec{v}_A - \vec{v}_1) + m_B(\vec{v}_B - \vec{v}_1) + m_C(\vec{v}_C - \vec{v}_1)$$

$$= -(m_1 + m_A + m_B + m_C)\vec{v}_1$$

$\therefore$
$$\vec{v}_1 = -\frac{\{m_A\vec{v}_{Arel} + m_B\vec{v}_{Brel} + m_C\vec{v}_{Crel}\}}{m_1 + m_A + m_B + m_C}$$

$$= -\frac{\{mv_0\hat{j} + (2mv_0\hat{i}) + 3m3v_0\cos 37\hat{i} + 3v_0\sin 37\hat{j}\}}{4m + m + 2m + 3m}$$

$$= -\frac{56}{50}\hat{i} - \frac{77}{50}\hat{j}$$

$\therefore$
$$|v_1| = \sqrt{\left(\frac{56}{50}\right)^2 + \left(\frac{77}{50}\right)^2}$$

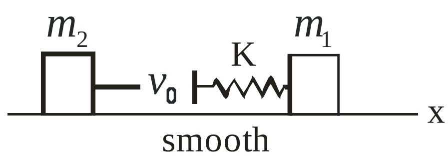
= 1.9 m/s

2.5 Motion of centre of mass with constant velocity

From eq^n 2.3(iii), $\sum \vec{F}ext = M\vec{a}_{cm}$, it is clear that if net external force on a system is zero, the acceleration of centre of mass is zero. It means centre of mass moves with constant velocity. The displacement of centre of mass of system in time *t* is

$$\vec{s} = \vec{v}_{cm}t.$$

Example 2.17



(Figure 2.17(A))

Figure 2.17(A) shows two blocks of masses m_1 and m_2 placed on smooth horizontal surface. At $t = 0$, the centre of mass of these two blocks system is at the origin. At $t = t_0$, the spring is in the state of maximum compression.

(A) Find momentum of system at $t = 0$ and $t = t_0$.

(B) Find position of centre of mass at $t = t_0$.

Solution (A) The initial velocity of centre of mass is

$$u_{cm} = \frac{m_2 v_0}{m_1 + m_2}$$

As net external force on system in horizontal direction during motion of blocks is zero, the velocity of centre of mass will remain constant.

$$\therefore \qquad V_{cm} = u_{cm}$$

$\because$ The momentum of system is

$$P = (m_1 + m_2) v_{cm}$$

$$= (m_1 + m_2) \frac{m_2 v_0}{m_1 + m_2}$$

$$= m_2 v_0$$

(B) $\because \qquad x - x_0 = v_{cm} t_0$

or $$x - 0 = \left(\frac{m_1 v_0}{m_1 + m_2}\right) t_0$$

∴ $$x = \frac{m_1 v_0 t_0}{m_1 + m_2}$$

Example 2.18

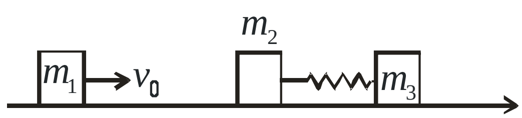


Figure 2.18(A)

Figure 2.18(A) shows three blocks of masses m_1, m_2 and m_3 placed on smooth horizontal surface. Masses m_2 and m_3 are connected by a light spring. Collision between m_1 and m_2 takes place at $t = t_0$.

(A) Find momentum of the system of blocks at $t = 2t_0$.

(B) Find displacement of the centre of mass of the system of blocks in time $2t_0$.

Solution No external force acts on the system of blocks. Therefore, the centre of mass of system moves with constant velocity.

∵ $$v_{cm} = \frac{m_1 v_1 + m_2 v_2 + m_3 v_3}{m_1 + m_2 + m_3}$$

$$= \frac{m_1 v_0}{m_1 + m_2 + m_3}$$

(A) The momentum of system is

$$P = (m_1 + m_2 + m_3) v_{cm}$$

$$= \frac{(m_1 + m_2 + m_3) m_1 v_0}{m_1 + m_2 + m_3}$$

$$= m_1 v_0$$

(B) The displacement of centre of mass of the system in time $2t_0$ is

$$S = v_{cm}(2t_0)$$

$$= \frac{2m_1 v_0}{m_1 + m_2 + m_3}$$

2.19 Two blocks A and B of masses m and 2m respectively and carrying charges $+q_1$ and $-q_2$ are kept at distance d apart on a smooth horizontal surface. Now block A is projected with speed v_0 away from the block B. Find velocity of centre of mass of two blocks system at an instant t.

Solution In this case, the weight of the blocks and the normal reaction balance each other. The force of electric interaction is internal force of the system. As net external force on the system is zero, the centre of mass of system moves with constant velocity

$$\because \quad v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$= \frac{m v_0}{m + 2m} = \frac{v_0}{3}$$

2.20 Particles A and B of masses m and 2m respectively are moving as such the velocity of the centre of mass of the system of blocks is $v_0 \hat{i}$. During motion, they interact to each other in complicated manner. At an instant t, the velocity of particle A is $a_0 \left(1 - e^{-bt}\right)\hat{i}$. Find the velocity of particle B at instant t.

Solution As velocity of centre of mass is constant, no net external force acts on the system.

$$\because \quad \vec{v}_{cm} = \frac{m_A\vec{v}_A + m_B\vec{v}_B}{m_A + m_B}$$

or
$$v_0\hat{i} = \frac{ma_o\left(1 - e^{-bt}\right)\hat{i} + 2m\vec{v}_B}{m + 2m}$$

or
$$3v_0\hat{i} = a_0\left(1 - e^{-bt}\right)\hat{i} + 2\vec{v}_B$$

or
$$2\vec{v}_B = 3v_0\hat{i} - a_0\left(1 - e^{-bt}\right)\hat{i}$$

$$\because \quad \vec{v}_B = \left\{\frac{3}{2}v_0 - \frac{a_0}{2}\left(1 - e^{-bt}\right)\right\}\hat{i}.$$

2.6 Accelerated motion of centre of mass

If net external force on a system is not zero, the acceleration of the centre of mass of the system is non zero and the velocity of the centre of mass changes. The acceleration of centre of mass is simply found by net external force on the system divided by total mass of the system. Let us consider some examples associated with accelerated motion of centre of mass.

2.21 Uniformly accelerated motion of centre of on a straight line

Two blocks of masses m_1 = 2 kg and m_2 = 3 kg lying on smooth horizontal surface. At t = 0, m_2 is given a velocity 10 m/s in leftward direction and a force of 10 N is applied on the block of mass m_1 in rightward direction (shown in **Figure 2.21(A)**).

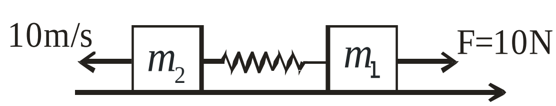


Figure 2.21(A)

(A) Find momentum of the two blocks system at t = 1 sec.

(B) Find velocity of centre of the two blocks system at t = 3 sec.

(C) Find displacement of centre of mass of the two blocks system from t = 0 to t = 2 sec.

Solution Net external force on the system is F = 10 N in rightward direction. Therefore the acceleration of the centre of mass of the system is

$$a_{cm} = \frac{F}{m_1 + m_2}$$

$$= \frac{10}{2+3} = 2m/s^2 \qquad \text{in rightward direction.}$$

(A) As acceleration of centre of mass is constant, kinematics equations are applicable. Take rightward direction as positive. The initial velocity of centre of mass is

$$u_{cm} = \frac{m_1u_1 + m_2u_2}{m_1 + m_2}$$

$$= \frac{-10\times 3}{2+3}$$

= - 6 m/s

$\because$ $$v_{cm} = u_{cm} + a_{cm}t$$

or $$v_{cm} = -6 + 2\times 1 = -4m/s.$$

The momentum of system at t = 1 sec is

$$P = (m_1 + m_2)v_{cm}$$

= - 20 kg m/s

Thus, the momentum of system at t = 1 s is 20 kg m/s in leftward direction.

(B) ∵ $v_{cm} = u_{cm} + a_{cm}\, t$

or $v_{cm} = -6 + 2 \times 3$

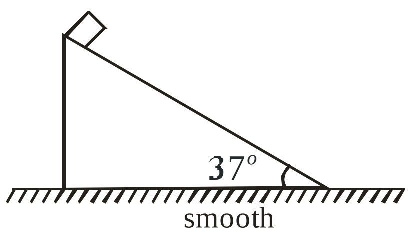
$= 0$

(C) $$s = u_{cm}t + \frac{1}{2}a_{cm}t^2$$

$$= -6 \times 2 + \frac{1}{2} \times 2 \times 2^2$$

$$= -12 + 4 = -8m.$$

2.22 A block of mass m_1 = 5 kg is released from rest from top of large triangular smooth block of mass m_2 = 10 kg (Shown in **Figure 2.22(A)**).



(Figure 2.22(A))

(A) Does centre of mass of blocks system move horizontally?

(B) Does the centre of mass of blocks system move on vertical straight line?

(C) Find acceleration of the centre of mass of the blocks system.

(D) Find momentum of the blocks system at *t* =1s.

Solution The external forces on the blocks system are weight of blocks, $(m_1 + m_2)$ g in downward direction and normal reaction 'N' by horizontal surface in upward direction.

(A) No,

As frictional force is absent, net external force in horizontal direction is zero. Therefore the acceleration of the centre of mass of the system in horizontal direction is zero.

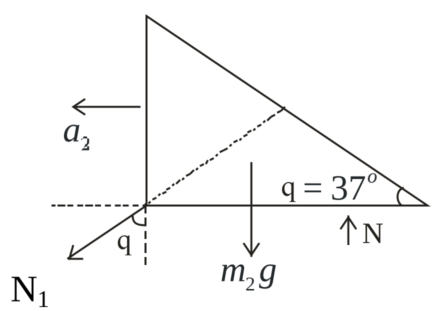
(B) Yes,

Initially, centre of mass is at rest. As the block of mass m_1 will move down along inclined surface, the centre of mass will start to move vertically downward. For this, weight of blocks, $(m_1 + m_2)$ g should be greater than the normal reaction N of horizontal surface. Therefore, the acceleration of the centre of mass of the system is

$$a_{cm} = \frac{(m_1 + m_2)g - N}{m_1 + m_2}$$

along vertical straight line in downward direction.

(C) From force diagram of the triangular block (shown in **Figure 2.22(B)**),



(Figure 2.22(B))

∵ $N_1 \sin\theta = m_2 a_2$

or $N_1 \sin 37 = 10 a_2$

∵ $$N_1 = \frac{10a_2}{\frac{3}{5}} = \frac{50}{3} a_2 \qquad \text{(i)}$$

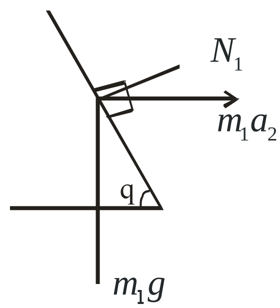
And $$N = m_2 g + N_1 \cos q$$

$$N = 100 + \frac{50}{3} a_2 \cos 37$$

or $$N = 100 + \frac{50}{3} a_2 \times \frac{4}{5}$$

$\because$ $$N = 100 + \frac{40}{3} a_2 \qquad \text{(ii)}$$

From force diagram of block of mass m_1 in the frame of triangular block (shown in **Figure 2.22(C)**)



(Figure 2.22(C)

$$N_1 + m_1 a_2 = \sin^{-} = m_1 g \cos^{-}$$

or $$N_1 = 50 \times \frac{4}{5} - 5a_2 \times \frac{3}{5}$$

or $$N_1 = 40 - 3a_2 \qquad \text{...(iii)}$$

From eq^n (i) and (iii),

$$\frac{50}{3} a_2 = 40 - 3a_2$$

or $$\frac{59}{3} a_2 = 40$$

$\because$ $$a_2 = \frac{120}{59}$$

From eq^n (ii),

$$N = 100 + \frac{40}{3} a_2$$

$$= 100 + \frac{40}{3} \times \frac{120}{59}$$

$$= 127.1 \text{ N}$$

$\because$

$$a_{cm} = \frac{(m_1 + m_2)g - N}{m_1 + m_2}$$

$$= \frac{150 - 127.1}{15} = 1.52 m/s^2$$

(D)

$$v_{cm} = u_{cm} + a_{cm}t$$

$$= 0 + 1.52\,'\,1$$

$$= 1.52 \text{ m/s}$$

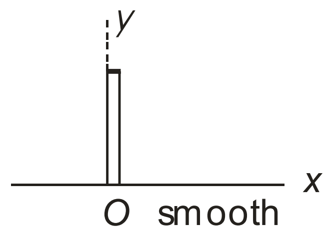
The momentum of system at t = 1s is

$$P = (m_1 + m_2)v_{cm}$$

$$= 15 \times 1.52 = 22.8 \text{ kg m/s}$$

Thus, the momentum of system at t = 1 sec is 22.8 kg m/s in downward direction.

2.23 **Figure 2.23(A)** shows of a uniform thin rod of mass m and length L placed vertically on a smooth horizontal surface. Now lower end starts to slip due to a slight jerk.



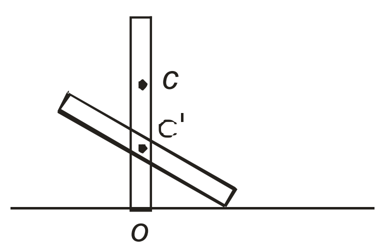
(Figure 2.23(A))

(A) Is trajectory of the centre of mass vertical straight line?

(B) Find the equation of trajectory of a point of rod at distance ℓ_0 from the lower end.

(A) Yes

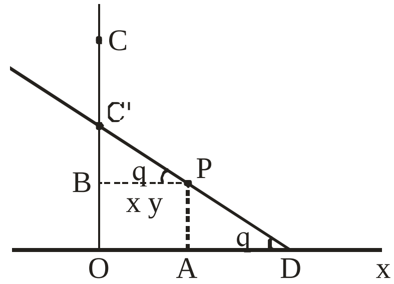
Initially centre of mass of rod is at rest. As surface is smooth, net external force on the rod in horizontal direction is zero. Therefore, the centre of mass of the rod does not move in horizontal direction. In vertical direction, external forces on the rod are the weight mg of rod in downward direction and normal reaction 'N' of horizontal surface in upward direction. Therefore centre of mass falls down vertically as shown in **Figure 2.23(B)**.



(Figure 2.23(B))

(B) At an instant t, the co-ordinates of point P are (x, y) as shown in figure 2.23(C).

From **Figure, 2.23(C)**,



Figure, 2.23(C)

$$\sin\theta = \frac{PA}{PD} = \frac{y}{\ell_0} \qquad \text{(i)}$$

Also, $$\sin\theta = \frac{C'O}{C'D}$$

$\therefore$ $$C'O = \frac{\ell}{2}\sin\theta \qquad \left(\because C'D \simeq \frac{\ell}{2}\right) \qquad \text{(ii)}$$

In $\Delta\ C'PB,$

$$\cos\theta = \frac{BP}{C'P} = \frac{x}{\frac{\ell}{2} - \ell_0} \qquad \text{(ii)}$$

From equation (i) and (ii),

$$\left(\frac{y}{\ell_0}\right)^2 + \left(\frac{x}{\frac{\ell}{2} - \ell_0}\right)^2 = \sin^2\theta + \cos^2\theta = 1$$

$\therefore$ $$\frac{x^2}{\left(\frac{\ell}{2} - \ell_0\right)^2} + \frac{y^2}{\ell_0^2} = 1$$

Thus, the trajectory of the point P is elliptical in shape.

If $\ell_0 = \frac{\ell}{4}$, the trajectory of point P is circular in shape

2.24. A triangular plate of mass m is placed on smooth horizontal surface. Three forces start to act as shown in **Figure 2.24(A)**. Find the initial acceleration of centre of mass of the triangular plate.

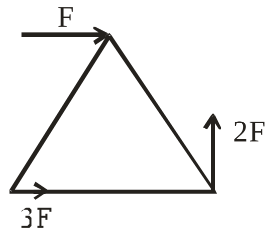


Figure 2.24(A)

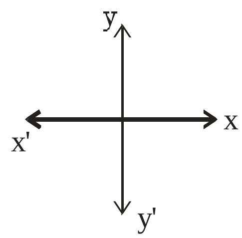


Figure 2.24(B)

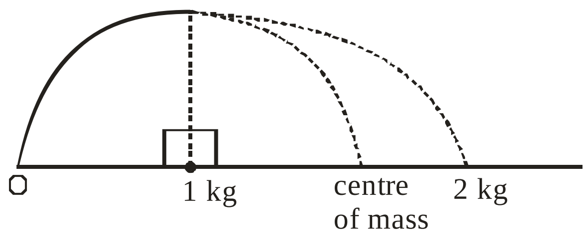
Solution The net external force on the plate is $\vec{\mathbf{F}}\mathbf{ext} = 4F\hat{i} + 2F\hat{j}$. The acceleration of centre of mass of the plate is

$$\vec{a}_{cm} = \frac{\vec{\mathbf{F}}\mathbf{ext}}{m}$$

$$= \frac{4F}{m}\hat{i} + \frac{2F}{m}\hat{j}.$$

2.25 A block of mass 3 kg is projected with a speed of 10 m/s at an angle 15^0 with horizontal. It explodes into two pieces of masses 1 kg and 2 kg at the top of its flight. The piece of 1 kg mass drops straight down from rest after the explosion while the piece of 2 kg mass moves off horizontally, so that they land simultaneously. Where does the piece of 2 kg mass land? (Neglect air resistance)

Solution Considering the block to be the system, the centre of mass continues on the same parabolic path as the block's path before exploding. The reason behind this is that the only external force before the pieces hit the ground is the force of gravity. The forces exerted in the explosion are internal forces,



these forces do not affect the motion of the centre of mass. The range of path followed by centre of mass is

$$R = \frac{u^2 \sin 2a}{g} = \frac{10^2 \sin 30}{10}$$

$$= 5 \text{ m.}$$

$\because$ $$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or $$5 = \frac{1 \times \frac{5}{2} + 2x_2}{1+2}$$

or $$15 - \frac{5}{2} = 2x_2$$

$\therefore$ $$x_2 = \frac{25}{4} m$$

$$= 6.25 \text{ m}$$

Thus, the piece of 2 kg will land at distance 6.25 m from the point of projection.

2.26 A man of mass 50 kg leaps vertically upward with initial velocity 10 m/s from a spring board. As he rises up, he takes off his son of mass 30 kg sitting at the horizontal branch of a tree at height 2 m from the spring board. What is the maximum height attained by the pair.

Solution Consider man and his son as a system.

The velocity of man just before catching the son is

$$v_0^2 = 10^2 - 2 \times 10 \times 2 = 60$$

$\therefore$ $$v_0 = \sqrt{60} m/s.$$

The velocity of centre of mass of the system just after catching the son by the man is

$$u_{cm} = \frac{50\sqrt{60}}{50+30} = \frac{5\sqrt{60}}{8} m/s$$

The maximum height attained by the combined system is

$$H = 2 + \frac{u_{cm}^2}{2g} = 2 + \frac{25}{64} \times \frac{60}{2 \times 10}$$

$$= 2 + 1.17 = 3.17 \text{ m.}$$

2.7 The C-Frame

The reference frame rigidly fixed to centre of mass of a system of particles is known as the centre of mass frame, or briefly the C-frame. Let us consider a system of n particles of total mass M = m_1 +m_2 +…+m_n. These particles are moving with velocities $\vec{v}_1, \vec{v}_2 ... \vec{v}_n$ respectively. The velocity of the centre of mass of the system is

$$\vec{v}_C = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 ... m_n\vec{v}_n}{m_1 + m_2 + ... + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{\sum_{i=1}^{n} m_i} \qquad (2.7(I))$$

The momentum of i^{th} particle in C-frame is

$$\vec{P}_{ic} = m_i\left(\vec{v}_i - \vec{v}_c\right)$$

$$= m_i\vec{v}_i - m_i\vec{v}_c$$

The total momentum of the system in C-frame is

$$\sum_{i=1}^{n}\vec{P}_{ic} = \sum_{i=1}^{n} m_i\vec{v}_i - \sum_{i=1}^{n} m_i\vec{v}_c$$

Putting the value of eq^n 2.7(i),

$$\sum_{i=1}^{n}\vec{P}_{ic} = \sum_{i=1}^{n} m_i\vec{v}_c - \sum_{i=1}^{n} m_i\vec{v}_c$$

$$= \vec{0}.$$

Thus, it is one of distinctive feature of C-frame that the total momentum of a system in c-frame is zero.

Once the momentum of system in C-frame calculated, it is easy to calculate the kinetic energy of system in C-frame.

The kinetic energy of ith particle in C-frame is

$$T_{ic} = \frac{1}{2}m_i v_{ic}^2$$

$$= \frac{1}{2}m_i\left(\vec{v}_i - \vec{v}_c\right)^2$$

$$= \frac{1}{2}m_i\left(v_i^2 + v_c^2 - 2\vec{v}_i\vec{v}_c\right)$$

$$= \frac{1}{2}m_i v_i^2 + \frac{1}{2}m_i v_c^2 - m_i\vec{v}_i\vec{v}_c$$

$\therefore$ The total kinetic energy of the system in C-frame is

$$T_{cm} = \sum_{i=1}^{n} T_{ic}$$

$$= \sum \frac{1}{2} m_i v_i^2 + \frac{1}{2} v_c^2 \sum m_i - \sum m_i \vec{v}_i . \vec{v}_c$$

$$= T + \frac{1}{2} M v_c^2 - M \sum \vec{v}_c . \vec{v}_c$$

$$= T + \frac{1}{2} M v_c^2 - M v_c^2$$

$$\therefore \quad T_{cm} = T - \frac{1}{2} M v_c^2$$

$$\therefore \quad T = T_{cm} + \frac{1}{2} M v_c^2 \qquad (2.7(ii))$$

Where T = kinetic energy of system

T_{cm} = kinetic energy of the system in C-frame

M = Total mass of the system.

From this, it is important to keep in mind that the kinetic energy of a system of particles is the sum of the kinetic energy in c-frame. T_{cm} and kinetic energy associated with the centre of mass motion, $\frac{1}{2} m v_{cm}^2$.

Total mechanical energy of a system consists of potential energy and kinetic energy of the system. The potential energy of a system is same in all reference frames while kinetic energy of a system is frame dependent.

Thus, the mechanical energy of a system is also frame dependent. Adding potential energy ‘U’ in both sides of equation 2.7(ii), we obtain the formula for mechanical energy of a system.

$\because$ $$T = T_{cm} + \frac{1}{2} M v_{cm}^2$$

or $$T + U = T_{cm} + U + \frac{1}{2} M v_{cm}^2$$

or $$E = E_{cm} + \frac{1}{2} M v_{cm}^2$$

Where E = Total mechanical energy of a system of particles

E_{cm} = Total mechanical energy of system in C- frame.

Total mechanical energy of a system in C-frame is also known as internal mechanical energy of the system.

It no external forces act on the system, the centre of mass of system moves with constant velocity.

The C-frame of such type of system is inertial in nature. If net external force on the system is non zero, C-frame is non inertial in nature.

2.8 C-frame of a system of two particles

Let us consider a system of two particles of masses m_1 and m_2 and their velocities are $\vec{v}_1$ and $\vec{v}_2$ respectively.

The momentum of first particle in C-frame is

$$\vec{P}_{1c} = m_1 \vec{v}_{1c}$$

$$= m_1 (\vec{v}_1 - \vec{v}_c)$$

$$= m_1 \left\{ \vec{v}_1 - \left(\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \right) \right\}$$

$$= \frac{m_1 m_2}{m_1 + m_2} (\vec{v}_1 - \vec{v}_2)$$

$$= m(\vec{v}_1 - \vec{v}_2) = m\vec{v}_{12}.$$

Where $m = \dfrac{m_1 m_2}{m_1 + m_2}$ = it is known as reduced mass of the system and $\vec{v}_{12}$ is relative velocity of first particle with respect to second particle.

Similarly, the momentum of second particle is

$$\vec{P}_{2c} = m(\vec{v}_2 - \vec{v}_1) = m\vec{v}_{21}.$$

Thus, the momenta of both particles in C-frame are equal in magnitude but opposite in direction.

i.e. $$\left|\vec{P}_{1c}\right| = \left|\vec{P}_{2c}\right| = m\left|\vec{v}_{rel}\right|$$

The kinetic energy of the system of two particles in C-frame is

$$T_{cm} = \frac{P_{1c}^2}{2m_1} + \frac{P_{2c}^2}{2m_2}$$

$$= \frac{\left(\text{«}\left|\vec{v}_{rel}\right|\right)^2}{2}\left(\frac{m_1 + m_2}{m_1 m_2}\right)$$

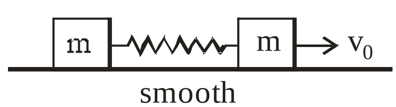
$$= \frac{m^2\left|\vec{v}_{rel}\right|^2}{2} , \frac{1}{m}$$

$\therefore$ $$T_{cm} = \frac{1}{2}m\left(\left|\vec{v}_{rel}\right|\right)^2$$

Thus, the total mechanical energy of system in C-frame is

$$E_{cm} = \frac{1}{2}m\left|\vec{v}_{rel}\right|^2 + U$$

2.27



(Fig. 2.27(A))

Two blocks each of mass m = 10 kg lying on a smooth horizontal surface are interconnected with a light spring of spring constant 100 N/m. One of the blocks is set in notion with speed 10 m/s at the instant when elongation in spring is 10 cm.

(A) Find internal mechanical energy of this system in the process of motion.

(B) Find magnitude of momentum of each particle in C-frame.

(C) Find total momentum of system in C-frame.

Solution (A) Since, the surface is smooth and spring force is conservative in nature. Therefore, the total mechanical energy of the system and the velocity of centre of mass remain constant. The kinetic energy of the system in C-frame is

$$T_{cm} = \frac{1}{2} m v_{rel}^2$$

$$= \frac{1}{2}\left(\frac{m \times m}{m + m}\right)(10 - 0)^2$$

$$= \frac{1}{2} \times \frac{10}{2} \times 100 = 250 J$$

The potential energy is

$$U = \frac{1}{2} k x^2$$

$$= \frac{1}{2} \times 100 \times (0.1)$$

= 0.5 J

Thus, the internal mechanical energy of system

= The mechanical energy of the system in C-frame

$= T_{cm} + U = 250 + 0.5$

$= 250.5$ Joule.

(B) The magnitude of momentum of each block in C-frame is

$$\left|\vec{P}_{1c}\right| = \left|\vec{P}_{2c}\right| = m\left|\vec{v}_{rel}\right|$$

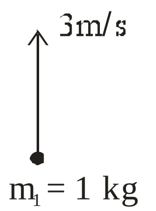
$$= \left(\frac{m \times m}{m+m}\right)(10-0)$$

$$= \frac{10}{2} \times 10$$

$= 50$ kg m/s

(C) Total momentum of the system in C-frame is zero.

2.28 Calculate the kinetic energy of the system of particles as (shown in **Figure 2.28(A)**)



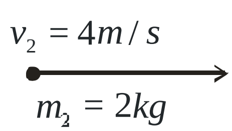


Figure 2.28(A) **Figure 2.28(B)**

Solution The kinetic energy of the system in C-frame is

$$T_{cm} = \frac{1}{2} m\left|\vec{v}_{rel}\right|^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\left(v_1^2 + v_2^2 - 2v_1 v_2 \cos 90\right)$$

$$= \frac{1}{2}\left(\frac{1 \times 2}{1+2}\right)\left(3^2 + 4^2\right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times 25 = \frac{25}{3} \quad \text{Joule}$$